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Long-wave/short-wave interactions in flow between concentric cylinders

By NICOLA J. HORSEMAN AND ANDREW P. BASSOM
Department of Mathematics, North Park Road, University of Exeter, Exeter, Devon, EX4 4QE, UK

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Consider the flow of an incompressible fluid between two infinite concentric circular cylinders. The outer cylinder is at rest whilst the angular velocity of the inner cylinder has a steady part and also a harmonically oscillating component. We examine the situation where, for a suitable choice of parameters, two types of vortex instability can occur simultaneously; first a short-wavelength mode which is essentially trapped in a thin 'Stokes' layer near the inner cylinder and, secondly, a long-wavelength mode which fills the whole region between the cylinders. We investigate the problem in which two short-wavelength vortices and one long-wavelength vortex coexist and are such that each pair interacts to drive the third. Additionally, the short-wavelength disturbances are nonlinear in their own right. Coupled amplitude equations for the three modes are derived and their solution discussed.

This form of interaction may also take place in a boundary layer. Such a situation is more complex than that under consideration here as it would be necessary to take into account the growth of the boundary layer. However, this simplified problem gives an insight into the behaviour of the more difficult situation.

1. Introduction

The aim of this paper is to give a theoretical description of a particular interaction phenomenon which can occur between vortices in an incompressible viscous fluid contained between two infinitely long concentric circular cylinders. In particular, we concentrate on the case of a modulated Taylor-vortex flow in which the outer cylinder is at rest whereas the inner one has an angular velocity which modulates about a non-zero mean. A review of Taylor-vortex flows in general has been given by Stuart (1986), but the modulated case has relevance in numerous practical situations and thus has merited special attention; examples include circulatory flows in animals and many geophysical considerations.

Some of the first experimental work on modulated flows between long concentric cylinders was performed by Donnelly (1964). He studied the flow in the geometry described above and found that the modulation of the steady part of the angular velocity of the inner cylinder by the oscillatory component markedly stabilized the flow. Hall (1975) gave an asymptotic analysis in an attempt to explain theoretically the results obtained by Donnelly and, to this end, used both linear and weakly nonlinear approaches. He took the angular velocity of the inner cylinder as $\Omega(1+\epsilon\cos\omega t)$ and analysed two particular limits. Firstly, for small ω and ϵ, it was found, according to linear theory, that the critical Taylor number at which the basic flow becomes prone on instability is $O(\epsilon^2)$ less than its value for the unmodulated case.

Additionally, in the large-frequency limit with ϵ arbitrary, it was shown that modulation again has the effect of decreasing the critical Taylor number, but now alters its value by $O(\epsilon^2\omega^{-3})$. In his study, Hall defined the Taylor number by $T = 2\Omega^2 R_1 d^3/\nu^2$, where R_1 is the radius of the inner cylinder, d is the gap between the cylinders and ν is the kinematic viscosity of the fluid. In the unmodulated case the critical value of T at which the onset of instability occurs is $T_c \approx 3390$. The work concentrated on the 'small gap limit', in which the distance between the cylinders is assumed to be much less than the radius of curvature of either cylinder and the perturbations investigated were taken to have axial wavelength comparable with the separation of the cylinders.

The analysis of Hall (1975) showed, in the high-frequency limit and on the basis of linear theory, that the vortex flow is governed by a multilayered structure. The basic flow in this situation is time-dependent only in a region close to the inner cylinder (essentially in a Stokes layer). Further from this boundary the basic flow attains an essentially steady form and the perturbation structure (which extends across the whole of the region between the cylinders) is correspondingly different from that in the Stokes layer. Finally, a third region, which takes the form of a Stokes layer adjacent to the outer cylinder, is required in order to ensure that the necessary boundary conditions are satisfied on that cylinder wall. In each of these zones, Hall expanded the disturbance velocity in a Fourier time series and then expanded the Fourier coefficients and the Taylor number T in powers of $\sigma^{-\frac{1}{2}}$, where σ is defined as the (large) frequency parameter $\omega d^2/\nu$. By equating like powers of $\sigma^{-\frac{1}{2}}$ and solving the resulting systems of differential equations within each zone, Hall determined the disturbance velocity across the whole region between the two cylinders. Further, the coefficients in the Taylor number expansion were found by examination of the steady parts of the velocity field.

Hall also investigated the problem for small finite-sized vortices in the high-frequency limit by utilizing a standard weakly nonlinear approach. The Taylor number was perturbed slightly from its linear neutral value and the amplitude of the vortex permitted to evolve on a suitably slow timescale. Hall (1975) demonstrated that the effect of nonlinearity is only important through its influence on the steady part of the perturbation velocity, and that, according to weakly nonlinear theory, small equilibrium vortices could persist. Additional details of all the aspects of the problem considered by Hall (1975) may be found in his thesis, Hall (1973).

The theoretical results of Hall (1975) for the high-frequency linear modes suggest that modulation tends to destabilize the flow; in contrast to the experimental conclusion reached by Donnelly (1964). Kumar, Bhattacharjee & Banerjee (1986) attempted to repeat the high-frequency analysis of Hall, but derived conclusions very different from those of the latter investigation. In particular, Kumar *et al.* suggested that modulation has a stabilizing influence and that it has a very small, $O(\omega^{-5})$, effect on the critical Taylor number. However, these authors failed to take any account whatsoever of the delicate physical structure governing the flow. More recently, experiments by Walsh, Wagner & Donnelly (1987) have lent support to the view that modulation does indeed destabilize the flow, reinforcing the conclusions reached by Hall.

In this paper, we shall consider interactions between a high-frequency vortex of the type described by Hall (1975, hereinafter referred to as H) and a pair of vortices of much shorter, but different, wavelengths, the structure of which was first considered by Seminara & Hall (1976, hereinafter SH). In SH the linear stability of a Stokes layer on a harmonically oscillating cylinder within an infinite, viscous, incompressible fluid was examined. At a large frequency of oscillation ω, the flow induced in the fluid by the motion of the cylinder is confined to a thin layer of thickness $O((\nu/\omega)^{\frac{1}{2}})$ adjacent to the cylinder. SH argued that since the flow is confined to this boundary layer, then at leading order the centripetal forces associated with the curvature of the streamlines have negligible effect on the basic flow, although this curvature ensures that the basic flow is susceptible to a vortex-type instability whose wavelength in the axial direction of the cylinder is also $O((\nu/\omega)^{\frac{1}{2}})$. In the Stokes-type layer, the disturbance velocity was expanded as a Fourier series in time, as in H, and this resulted in an infinite-dimensional system of ordinary differential equations. The perturbations decay exponentially at the edge of the Stokes layer and were evaluated using two distinct methods. Firstly, SH demonstrated that the solution of these governing differential equations could be obtained using an analytical approach. By seeking a solution of Hill type it was shown that the perturbation velocities could be determined in the form of infinite sums. The summands are found by solving an infinite, homogeneous system of linear algebraic equations which necessitates the location of zeros of an infinite determinant.

It was discovered that in practice this analytic solution was difficult to obtain. Consequently, SH also examined a numerical solution of the differential system. The details of this approach are very similar to those of the method we use later in this paper and so we postpone a discussion of these numerical aspects until then. For the present, we remark that the numerical solution described by SH indicates that the flow between the concentric cylinders in H is liable to an instability of these small-wavelength vortices confined to the vicinity of the inner cylinder when the Taylor number T exceeds the value $164\epsilon^{-2}\sigma^{\frac{3}{2}}$. Here (in H), we recall that the angular velocity of the inner cylinder is $\Omega(1+\epsilon\cos\omega t)$ and σ is defined as the frequency parameter $\omega d^2/\nu$, where d is the gap between the cylinders. Noting the critical Taylor number of 3390 derived in H, at which the flow becomes unstable to long-wavelength vortices which completely fill the space between the cylinders, we see that when $\epsilon \approx 0.22\sigma^{\frac{3}{4}}$ the breakdown of the flow via either vortex type becomes equally likely, as mentioned by SH. At this crucial size for ϵ we can show that for a Taylor number very slightly above critical there are two short-wavelength vortices which can interact through the nonlinear terms in the Navier–Stokes equations and drive a longer wavelength mode. In turn, this long-wavelength perturbation can combine with each of the short-wavelength vortices to drive the other. It is this interaction mechanism which is the subject of our study here and, for ease of reference, we shall refer to the long-wavelength vortices described in H and the high-wavenumber modes of SH as being long-wave and short-wave perturbations respectively.

SH also conducted some simple experiments to confirm their theoretical predictions as to the critical Taylor number and the nature of the breakdown of the flow to short-wave vortices. In a subsequent paper, Seminara & Hall (1977), they extended their theory to a weakly nonlinear regime. Here, they used the method of multiple scales to monitor the nonlinear evolution of a monochromatic disturbance to the basic flow when the Taylor number T was slightly above or slightly below the linear critical value T_c. By using an approach similar to that employed by Stewartson & Stuart (1971), the authors sought disturbances of amplitude $O(|T-T_c|^{\frac{1}{2}})$ when $|T-T_c| \ll 1$. Seminara & Hall (1977) showed that according to this weakly nonlinear theory the nonlinear effects permit the existence of a non-zero, stable disturbance when linear theory would suggest instability and exponential growth of the perturbation. Duck (1979) considered the effect on this instability mode when the cylinder wall was made slightly wavy. In particular, he examined flows at Taylor numbers close to critical and a variety of wavelengths of the distortion in the cylinder, all of which had the property that the forcing caused by the wall perturbation interacted with modes of certain critical wavelengths so as to produce resonance. Hall (1981) further investigated the secondary breakdown of the weakly nonlinear finite-amplitude disturbances of Seminara & Hall (1977) to a mode with wavelength twice that of the fundamental vortex. The theoretical onset of this secondary breakdown was in good agreement with the results of experimental work by Park & Donnelly (1981).

In the interaction process between the long-wave and two short-wave vortices considered in our present work we allow the vortices to evolve on a suitably slow timescale which enables us to derive a coupled triple of amplitude equations for the modes. We find that it is possible to obtain an asymptotic description of the physical processes involved when the short-wave vortices are nonlinear in their own right (in the sense of Seminara & Hall 1977). The resulting triad of equations are somewhat modified versions of the classical types which govern numerous interactions in hydrodynamics. Many examples of the types of interacting triad equations and their corrresponding solutions which do occur in fluid mechanics may be found in Craik (1985) and the references therein.

The procedure for the remainder of the paper is as follows. In §2 we formulate the problem and determine the sizes of the modes which enable the desired interaction mechanisms to occur. We develop the perturbation expansions and obtain the coupled evolution equations in §3 and, in §4, describe the numerical work necessary for the evaluation of the coefficients in these equations. Finally, in §5 we obtain some solutions of the triad equations, study the stability of the equilibrium solutions and conclude with some discussion and suggestions for further work.

2. Formulation of the interaction problem

We consider the stability of a viscous incompressible fluid contained between two infinitely long concentric circular cylinders of radii R_1 and R_2 ($> R_1$). The gap d between the cylinders is taken to be small compared with R_1 and so in the ensuing analysis terms of $O(d/R_1)$ are neglected. The fluid is supposed to have density ρ and kinematic viscosity ν, and cylindrical polar coordinates (r, θ, Z) are taken with the Z-axis aligned along the axis of the cylinders. Let the corresponding velocity vector be (u^*, v^*, w^*) and t^* be the time, where an asterisk denotes a dimensional quantity.

Following the discussion contained in the introduction we suppose that the inner cylinder rotates about its axis with angular velocity

$$\Delta\omega/R_1\left(\frac{1}{\epsilon}+\cos\omega t^*\right),$$

where ω is the frequency of the oscillatory part of the angular velocity and Δ and ϵ are constants. However, we have already seen that when $\epsilon = O(\sigma^{\frac{3}{4}})$, where $\sigma = \omega d^2/\nu$ is the (assumed asymptotically large) frequency parameter, then the flow is susceptible to both types of instability mode described in §1.

The Stokes-type short-wavelength modes considered in SH are confined to a thin layer of thickness $O((\nu/\omega)^{\frac{1}{2}})$ adjacent to the inner cylinder. Since the work of SH shows that as we move away from this thin layer these short-wave modes decay exponentially, it may be anticipated that all the interactions between the different vortices take place inside the Stokes layer. Following SH we define dimensionless variables for this layer by

$$t = \omega t^*, \quad \eta = \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}(r-R_1), \quad z = \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}} Z, \tag{2.1}$$

and then the basic flow is given by $(0, \Delta\omega\bar{u}_0, 0)$, where

$$\bar{u}_0 = \frac{1}{\epsilon}\left[1-\eta\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\right] + \tfrac{1}{2}(e^{-\eta(1+i)+it} + \text{c.c.}), \tag{2.2}$$

and throughout the paper c.c. denotes complex conjugate. We suppose that this basic flow is perturbed so that the total flow has velocity components

$$(u(2\nu\omega)^{\frac{1}{2}}, \Delta\omega(v+\bar{u}_0), (2\nu\omega)^{\frac{1}{2}}).$$

Substituting these velocity expansions into the momentum and continuity equations and eliminating the pressure terms yields the governing sets of equations

$$\left(\mathrm{M}-2\frac{\partial}{\partial t}\right)\mathrm{M}u + T_s\,\bar{u}_0\frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_2}{\partial\eta\,\partial z}, \tag{2.3a}$$

$$\left(\mathrm{M}-2\frac{\partial}{\partial t}\right)v - 2u\frac{\partial\bar{u}_0}{\partial\eta} = Q_3, \tag{2.3b}$$

$$\frac{\partial u}{\partial\eta} + \frac{\partial w}{\partial z} = 0, \tag{2.3c}$$

where $\mathrm{M} \equiv (\partial^2/\partial\eta^2) + (\partial^2/\partial z^2)$, the Taylor number

$$T_s \equiv 2\Delta^2(2\omega)^{\frac{1}{2}}/(R_1\nu^{\frac{1}{2}}), \tag{2.3d}$$

and the nonlinear terms Q_1, Q_2 and Q_3 are defined by

$$\left.\begin{aligned} Q_1 &= 2\left(u\frac{\partial u}{\partial\eta} + w\frac{\partial u}{\partial z}\right) - \tfrac{1}{2}T_s v^2, \quad Q_2 = 2\left(u\frac{\partial w}{\partial\eta} + w\frac{\partial w}{\partial z}\right), \\ Q_3 &= 2\left(u\frac{\partial v}{\partial\eta} + w\frac{\partial v}{\partial z}\right). \end{aligned}\right\} \tag{2.3e}$$

We note that the definition of T_s corresponds to that given in Seminara & Hall (1977) but is $\surd 2$ times larger than the definition in SH. With these scalings, using the results of the introduction, the basic flow (2.2) is equally prone to the onset of the two types of instability mode when $\epsilon = \epsilon_0\sigma^{\frac{3}{4}}$, where $\epsilon_0 \approx (164/3390)^{\frac{1}{2}} \approx 0.220$. Finally, the no-slip conditions on the wall of the inner cylinder require that

$$u = v = w = 0 \quad \text{on} \quad \eta = 0. \tag{2.3f}$$

The short-wave modes have an $O(1)$ wavelength based upon the non-dimensionalization (2.1) and we demonstrate that we can formulate the interaction problem involving a long-wave vortex in which the Stokes modes investigated in SH are nonlinear in their own right. Using the results of H, the long-wavelength mode has wavenumber $O(\sigma^{-\frac{1}{2}})$ relative to the z-coordinate defined in (2.1) and its structure extends throughout the gap between the cylinders. If this vortex has z-dependence described by $E = \exp(ikz/\sigma^{\frac{1}{2}})$, where $k = O(1)$, and if δ is the (assumed small)

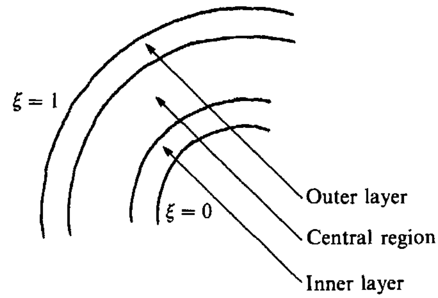

FIGURE 1. The flow structure for the long-wavelength (long-wave) mode. Here, $\eta = O(1)$ in the inner layer (the region to which short-wave vortices are confined), $\xi = \eta(2/\sigma)^{\frac{1}{2}} = O(1)$ in the central region described by $0 < \xi < 1$, and $\bar{\eta} = (\sigma/2)^{\frac{1}{2}} - \eta = O(1)$ in the Stokes layer adjacent to the outer cylinder.

amplitude of this disturbance mode, then in the 'inner layer' where $\eta = O(1)$, see figure 1, the leading-order perturbation velocities corresponding to this mode take the forms

$$\delta(\hat{u}, \sigma^{\frac{1}{4}}\hat{v}, \sigma^{\frac{1}{2}}\hat{w})\,E, \tag{2.4}$$

where $\hat{u}$, $\hat{v}$ and $\hat{w}$ are $O(1)$ functions of η and t. A feature of this mode is that the steady components of the disturbance velocities are asymptotically larger than their time-dependent parts. The disturbance does not decay as $\eta \rightarrow \infty$ and, indeed, H shows that it is largest in the central region of the flow where $\xi = \eta(2/\sigma)^{\frac{1}{2}} = O(1)$. In this zone, the basic flow solution (2.2) is steady and the disturbance quantities are governed by the solutions of a sixth-order system of ordinary differential equations with suitable boundary conditions at the edges of the channel, $\xi = 0, 1$. The flow structure for this vortex is completed by a thin layer adjacent to the outer cylinder. Details concerning this region may be found in H but to the orders we require in the current study careful analysis of this outer Stokes layer is rendered unnecessary.

Following SH we suppose that the critical Taylor number at which the flow is unstable to short-wave vortices is T_{sc} ($= 232.52$) with the corresponding wavenumber for the vortex m_c ($= 0.85852$). Then we consider the interaction of the two Stokes modes with axial dependences of the forms

$$E_{1,2} = \exp\left(\mathrm{i}(m_c \pm a_1 \sigma^{-\frac{1}{2}} + a_2 \sigma^{-1} + \ldots)\, z\right), \tag{2.5a}$$

where $a_2, a_3, \ldots$ are constants which need not be explicitly found. These two modes interact through the nonlinear terms Q_i in (2.3) to drive vortices with z-dependence of the form

$$E_3 = \exp\left(2\mathrm{i}a_1 \sigma^{-\frac{1}{2}} z\right), \tag{2.5b}$$

i.e. a long-wave mode. However, we have chosen T_{sc} and the size of the oscillatory part of the angular velocity of the inner cylinder so that the basic flow (2.2) is equally prone to instability through both long-wave and short-wave vortices and so we choose a_1 such that the mode with spatial dependence (2.5b) is neutrally stable at $T = T_{sc}$ and $\epsilon = \epsilon_0 \sigma^{\frac{3}{4}}$. Using the results of H this requires $a_1 \approx 2.2107$.

Guided by the work of Seminara & Hall (1977) we can show that when the amplitude of the Stokes modes is $O(\sigma^{-m})$, where m is some positive constant, these vortices are weakly nonlinear when allowed to evolve on a timescale of size $O(\sigma^{-2m})$. We then choose m and the amplitude of the long-wave vortex such that each pair of modes interact to drive the third. The Stokes modes combine in the inner layer where

$\eta = O(1)$ and the effect of this interaction on the long-wavelength mode manifests itself by altering the boundary conditions at $\xi = 0$ for the differential system governing the latter vortex in the central region. Additionally, this vortex interacts with the Stokes modes also inside the inner zone and by making a careful and thorough study of the governing equations (2.3) together with the forms of (2.4) and (2.5), it is found that the desired interaction coupling occurs when $m = 1$ and $\delta = O(\sigma^{-\frac{5}{4}})$. Then the modes evolve on a slow $O(\sigma^{-2})$ timescale and the amplitudes of the Stokes modes in the inner region are $O(\sigma^{-1})$. With this choice of parameters it is formally straightforward to verify that the long-wavelength mode remains essentially linear in character and all self-interactions involving this mode lead to terms which are negligible to our order of working.

To enable us to develop evolution equations for the three modes we further consider a Taylor number T_s slightly perturbed from its critical value so that

$$T_s = T_0 + \sigma^{-1}T_1 + \sigma^{-\frac{3}{2}}T_2 + \sigma^{-2}T_3 + \dots, \tag{2.6a}$$

where $T_0 = T_{sc}$, T_1 and T_2 are particular constants which will not need to be evaluated and T_3 remains unspecified for the moment. Further, we expand the steady part of the angular velocity of the inner cylinder as

$$\frac{1}{\epsilon} = \sigma^{-\frac{3}{4}}(\epsilon_1 + \sigma^{-\frac{1}{2}}\epsilon_2 + \sigma^{-1}\epsilon_3 + \dots), \tag{2.6b}$$

where $\epsilon_1 = 1/\epsilon_0 \approx 4.541$. The expressions (2.6) enable us to ensure that the modes remain neutrally stable at leading orders. Physically, the value of ϵ may be perturbed from its critical value by varying the size of the mean part of the angular velocity of the inner cylinder, with an increase in the velocity component corresponding to a reduction in ϵ. Additionally, inspection of (2.3d) reveals that the Taylor number T_s can be altered within an experimental context by changing the amplitude of the oscillation of the inner cylinder, by adjusting the frequency of the oscillatory part of its angular velocity, or, perhaps most easily, by changing the fluid contained within the gap between the cylinders. The perturbation velocities corresponding to the expansions (2.6) are formally considered in the following section and there we show how the interaction equations for the three modes may be derived.

3. Derivation of the evolution equations

To investigate the proposed interaction of the modes with z-dependencies given by (2.5) we follow the work contained in H, SH, Seminara & Hall (1977) and the discussion in the previous section. These accounts suggest that in the inner Stokes layer, where $\eta = O(1)$, the appropriate perturbation velocities assume the form

$$\begin{aligned} \boldsymbol{u} = {} & \sigma^{-1}[A(u_1, v_1, w_1)E_1 + B(u_2, v_2, w_2)E_2 + \text{c.c.}] \\ & + \sigma^{-\frac{5}{4}}[C(u_3, \sigma^{\frac{1}{4}}v_3, \sigma^{\frac{1}{2}}w_3)E_3 + \text{c.c.}] + \sigma^{-2}(|A|^2 + |B|^2)(u_m, v_m, w_m) \\ & + \sigma^{-2}[A^2(u_{21}, v_{21}, w_{21})E_1^2 + B^2(u_{22}, v_{22}, w_{22})E_2^2 + 2AB(u_{12}, v_{12}, w_{12})E_1E_2 + \text{c.c.}] \\ & + \dots, \end{aligned} \tag{3.1}$$

and that the modes evolve on an $O(\sigma^{-2})$ timescale, $\tau = \sigma^{-2}t$. In (3.1) we thus let the (possibly complex) amplitudes A, B and C of the three modes be functions of τ and the disturbance quantities $u_1, v_1, w_1, \dots, u_m, v_m, \dots, u_{21}, \dots$ are $O(1)$ functions of σ, η and t. The detailed expansions of the individual modes arise from the work of H and

SH, the corresponding amplitudes from considerations described in the preceding section and the forms of the mean flow and higher harmonic disturbances come from Seminara & Hall (1977). We use the technique of multiple scales to replace the time derivatives in (2.3) according to the rule

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}+\sigma^{-2}\frac{\partial}{\partial\tau}, \tag{3.2}$$

and, following SH, we expand the perturbation velocities as Fourier series in time which assume the form

$$(u_k, v_k, w_k) = \sum_{r=-\infty}^{\infty}(u_{kr}(\eta,\sigma), v_{kr}(\eta,\sigma), w_{kr}(\eta,\sigma))\,\mathrm{e}^{\mathrm{i}rt}, \tag{3.3a}$$

where k can take any of the subscript values 1, 2, 3, m, 21, 22 or 12 contained in (3.1). Then we express u_{kr}, v_{kr}, w_{kr} as descending series in the asymptotically large parameter σ so that

$$(u_{kr}, v_{kr}, w_{kr}) = (u_{kr0}(\eta), v_{kr0}(\eta), w_{kr0}(\eta))+\sum_{j=1}^{\infty}\sigma^{-\frac{1}{4}(1+j)}(u_{krj}(\eta), v_{krj}(\eta), w_{krj}(\eta)), \tag{3.3b}$$

and we can proceed to insert (2.5), (2.6), (3.1), (3.2) and (3.3) into the basic governing equations (2.3). Recalling the definitions of E_1, E_2 and E_3 in (2.5) we note that $E_1 = E_2E_3, E_2 = E_1E_3^{-1}, E_3 = E_1E_2^{-1}$ and these relations provide the mechanisms for the interactions between each pair of modes which drive the third. Much of the work described below can be derived from H and SH and, where possible, we merely state the governing equations for the perturbation quantities and appeal to H and SH for the corresponding solutions. The development of the amplitude equation for the second Stokes mode follows very closely that for the first such mode and so we concentrate on deriving the evolution equation for the latter vortex and indicate how it needs to be modified in order to obtain the analogous equation for the second mode.

Equating terms of leading order in σ with z-dependence proportional to E_1 leads to the coupled equations

$$\left.\begin{aligned}(\mathrm{L}-2\mathrm{i}r)\,\mathrm{L}u_{1(r)0}-\tfrac{1}{2}T_0\,m_c^2\,[\mathrm{e}^{-\eta(1+\mathrm{i})}v_{1(r-1)0}+\mathrm{e}^{-\eta(1-\mathrm{i})}v_{1(r+1)0}] = 0,\\ (\mathrm{L}-2\mathrm{i}r)\,v_{1(r)0}+(1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})}u_{1(r-1)0}+(1-\mathrm{i})\,\mathrm{e}^{-\eta(1-\mathrm{i})}u_{1(r+1)0} = 0,\end{aligned}\right\} \tag{3.4}$$

where $r = 0, \pm 1, \pm 2, \ldots,$ $\mathrm{L}\equiv(\mathrm{d}^2/\mathrm{d}\eta^2)-m_c^2$ and appropriate boundary conditions are that

$$u_{1(r)0}=\frac{\mathrm{d}u_{1(r)0}}{\mathrm{d}\eta}=v_{1(r)0}=0 \quad\text{at}\quad \eta=0 \quad\text{and}\quad u_{1(r)0},\ \frac{\mathrm{d}u_{1(r)0}}{\mathrm{d}\eta}\quad\text{and}\quad v_{1(r)0}\rightarrow 0 \quad\text{as}\quad \eta\rightarrow\infty.$$

This infinite system separates into two decoupled systems and, following SH, we take $u_{1(2r+1)0} = v_{1(2r)0} \equiv 0$. The remaining system for $(u_{1(2r)0}, v_{1(2r+1)0})$ is as stated and solved in Seminara & Hall (1977), i.e.

$$\left.\begin{aligned}(\mathrm{L}-4\mathrm{i}r)\,\mathrm{L}u_{1(2r)0}-\tfrac{1}{2}T_0\,m_c^2[\mathrm{e}^{-\eta(1+\mathrm{i})}v_{1(2r-1)0}+\mathrm{e}^{-\eta(1-\mathrm{i})}v_{1(2r+1)0}] = 0,\\ (\mathrm{L}-4\mathrm{i}r-2\mathrm{i})\,v_{1(2r+1)0}+(1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})}u_{1(2r)0}+(1-\mathrm{i})\,\mathrm{e}^{-\eta(1-\mathrm{i})}u_{1(2r+2)0} = 0,\end{aligned}\right\} \tag{3.5}$$

$r = 0, \pm 1, \pm 2, \ldots$, and

$$u_{1(2r)0} = \frac{\mathrm{d}u_{1(2r)0}}{\mathrm{d}\eta} = v_{1(2r+1)0} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty.$$

This system describes the behaviour of the fundamental components of the first Stokes mode and we find that the same coupled equations govern the equivalent terms for the other Stokes mode so that $u_{2(r)0}(\eta) = u_{1(r)0}(\eta)$ and $v_{2(r)0}(\eta) = v_{1(r)0}(\eta)$ for all r.

At first order we find that $v_{1(2r)1}(\eta) = u_{1(2r+1)1}(\eta) = 0$ and that the remaining functions $v_{1(2r+1)1}(\eta)$ and $u_{1(2r)1}(\eta)$ satisfy

$$\begin{aligned}(\mathrm{L}-4\mathrm{i}r)\,\mathrm{L}u_{1(2r)1} - \tfrac{1}{2}T_0\, m_c^2[\mathrm{e}^{-\eta(1+\mathrm{i})}v_{1(2r-1)1} + \mathrm{e}^{-\eta(1-\mathrm{i})}v_{1(2r+1)1}] \\ = T_0\, m_c\, a_1[\mathrm{e}^{-\eta(1+\mathrm{i})}v_{1(2r-1)0} + \mathrm{e}^{-\eta(1-\mathrm{i})}v_{1(2r+1)0}] + 4m_c\, a_1(\mathrm{L}-2\mathrm{i}r)\, u_{1(2r)0},\end{aligned} \tag{3.6a}$$

$$(\mathrm{L}-4\mathrm{i}r-2\mathrm{i})\, v_{1(2r+1)1} + (1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})}u_{1(2r)1} + (1-\mathrm{i})\,\mathrm{e}^{-\eta(1-\mathrm{i})}u_{1(2r+2)1} = 2m_c\, a_1\, v_{1(2r+1)0}, \tag{3.6b}$$

with $$u_{1(2r)1} = \frac{\mathrm{d}u_{1(2r)1}}{\mathrm{d}\eta} = v_{1(2r+1)1} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty.$$

It is also found that $u_{2(r)1}(\eta) = -u_{1(r)1}(\eta)$, $v_{2(r)1}(\eta) = -v_{1(r)1}(\eta)$ for all r. We show that we need the solutions of (3.6) and those of the next order system to enable us to describe the interaction of the two Stokes-layer modes which drives the long-wavelength vortex. This next order system is

$$v_{1(2r+1)2}(\eta) = u_{1(2r)2}(\eta) = 0, \tag{3.7a}$$

$$(\mathrm{L}-4\mathrm{i}r-2\mathrm{i})\,\mathrm{L}u_{1(2r+1)2} - \tfrac{1}{2}T_0\, m_c^2[\mathrm{e}^{-\eta(1+\mathrm{i})}v_{1(2r)2} + \mathrm{e}^{-\eta(1-\mathrm{i})}v_{1(2r+2)2}] = T_0\, e_1\, v_{1(2r+1)0}, \tag{3.7b}$$

$$(\mathrm{L}-4\mathrm{i}r)\, v_{1(2r)2} + (1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})}u_{1(2r-1)2} + (1-\mathrm{i})\,\mathrm{e}^{-\eta(1-\mathrm{i})}u_{1(2r+1)2} = 0, \tag{3.7c}$$

with $$u_{1(2r+1)2} = \frac{\mathrm{d}u_{1(2r+1)2}}{\mathrm{d}\eta} = v_{1(2r)2} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty.$$

Finally, $u_{2(r)2}(\eta) = u_{1(r)2}(\eta)$, $v_{2(r)2}(\eta) = v_{1(r)2}(\eta)$ for all r.

We now turn to consider the leading-order disturbance quantities for the long-wavelength mode. Substitution of the relevant terms of (3.1) in (2.3) yields that, as in H, we have

$$\frac{\mathrm{d}^4 u_{300}}{\mathrm{d}\eta^4} = 0, \quad \frac{\mathrm{d}^2 v_{300}}{\mathrm{d}\eta^2} = 0. \tag{3.8a}$$

Subject to the necessary boundary conditions $u_{300} = \mathrm{d}u_{300}/\mathrm{d}\eta = v_{300} = 0$ at $\eta = 0$, and setting one constant (obtained on integrating (3.8a)) equal to zero (which merely reduces the subsequent algebra, see H, rather than changing the final results), these equations have solutions

$$u_{300} = \hat{F}_1\,\eta^2, \quad v_{300} = \hat{F}_2\,\eta, \tag{3.8b}$$

and in turn, from (2.3c),

$$w_{300} = \frac{\mathrm{i}\hat{F}_1\,\eta}{a_1}. \tag{3.8c}$$

Here $\hat{F}_1$ and $\hat{F}_2$ are constants which may be determined by matching with the perturbation quantities in the central region, see §3.1.

As previously mentioned the time-dependent parts of this long-wavelength mode are much smaller than the steady components in the inner layer. In practice we have $u_{310} = v_{310} = w_{310} = 0$ and higher harmonic terms are yet smaller. In H, Hall has given details concerning these non-steady terms. It may be verified that all such terms are too small to have any effect on the leading-order interaction equations under consideration here and hence further analysis of these terms is not needed.

The first-order steady perturbation velocities for the long-wavelength mode in the inner zone are found to be dependent upon the low-order Stokes mode disturbances satisfying the differential systems (3.5)–(3.7). The two-short wavelength vortices interact through the nonlinear terms Q_j in the governing equations (2.3) to drive the long-wavelength vortex and we find that the steady perturbation terms u_{301} and v_{301} in (3.3b) are given by

$$\frac{\mathrm{d}u_{301}}{\mathrm{d}\eta^4} = I_{301}, \quad \frac{\mathrm{d}^2 v_{301}}{\mathrm{d}\eta^2} = J_{301}, \tag{3.9a}$$

where
$$I_{301} = I^{\dagger}_{301}\frac{A\bar{B}}{C}, \quad J_{301} = J^{\dagger}_{301}\frac{A\bar{B}}{C},$$

with
$$\begin{aligned} I^{\dagger}_{301} = &-4a_1^2 \sum_{r=-\infty}^{\infty} 4\frac{\mathrm{d}}{\mathrm{d}\eta}(u_{1(2r)0}\bar{u}_{2(2r)0}) \\ &-4a_1^2 \sum_{r=-\infty}^{\infty} T_0 v_{1(2r+1)0}\bar{v}_{2(2r+1)0} + \frac{8a_1^2}{m_c^2}\sum_{r=-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\mathrm{d}u_{1(2r)0}}{\mathrm{d}\eta}\frac{\mathrm{d}\bar{u}_{2(2r)0}}{\mathrm{d}\eta}\right) \\ &-\frac{4a_1^2}{m_c}\sum_{r=-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(u_{1(2r)0}\frac{\mathrm{d}^2\bar{u}_{2(2r)0}}{\mathrm{d}\eta^2} + \frac{\mathrm{d}^2 u_{1(2r)0}}{\mathrm{d}\eta^2}\bar{u}_{2(2r)0}\right) \\ &+\frac{4a_1}{m_c}\sum_{r=-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\mathrm{d}^2 u_{1(2r)1}}{\mathrm{d}\eta^2}\bar{u}_{2(2r)0} + \bar{u}_{2(2r)1}\frac{\mathrm{d}^2 u_{1(2r)0}}{\mathrm{d}\eta^2}\right. \\ &\left.-u_{1(2r)0}\frac{\mathrm{d}^2\bar{u}_{2(2r)1}}{\mathrm{d}\eta^2} - u_{1(2r)1}\frac{\mathrm{d}^2\bar{u}_{2(2r)0}}{\mathrm{d}\eta^2}\right), \end{aligned} \tag{3.9b}$$

and
$$\begin{aligned} J^{\dagger}_{301} = 2\sum_{r=-\infty}^{\infty}\Bigg(&u_{1(2r)0}\frac{\mathrm{d}\bar{v}_{2(2r)2}}{\mathrm{d}\eta} + u_{2(2r)0}\frac{\mathrm{d}\bar{v}_{1(2r)2}}{\mathrm{d}\eta} + u_{1(2r+1)2}\frac{\mathrm{d}\bar{v}_{2(2r+1)0}}{\mathrm{d}\eta} \\ &+u_{2(2r+1)2}\frac{\mathrm{d}\bar{v}_{1(2r+1)0}}{\mathrm{d}\eta} + v_{2(2r)2}\frac{\mathrm{d}\bar{u}_{1(2r)0}}{\mathrm{d}\eta} + v_{1(2r)2}\frac{\mathrm{d}\bar{u}_{2(2r)0}}{\mathrm{d}\eta} \\ &+v_{2(2r+1)0}\frac{\mathrm{d}\bar{u}_{1(2r+1)2}}{\mathrm{d}\eta} + v_{1(2r+1)0}\frac{\mathrm{d}\bar{u}_{2(2r+1)2}}{\mathrm{d}\eta}\Bigg). \end{aligned} \tag{3.9c}$$

Here a bar on a quantity denotes the complex conjugate of that quantity, and the components contained in the definitions (3.9b, c) are given by the systems (3.5)–(3.7). The terms I_{301}, J_{301} become exponentially small as $\eta \to \infty$ so that integrating (3.9a) yields the solutions

$$u_{301} = \hat{F}_3\eta^2 + \hat{F}_4\eta^3 + \tfrac{1}{2}\eta\int_0^\eta t^2 I_{301}(t)\,\mathrm{d}t - \tfrac{1}{6}\int_0^\eta t^3 I_{301}(t)\,\mathrm{d}t + \tfrac{1}{2}\eta^2\int_\eta^\infty t I_{301}(t)\,\mathrm{d}t - \tfrac{1}{6}\eta^3\int_\eta^\infty I_{301}(t)\,\mathrm{d}t, \tag{3.10a}$$

$$v_{301} = \hat{F}_5\eta + \int_0^\eta t J_{301}(t)\,\mathrm{d}t - \eta\int_\eta^\infty J_{301}(t)\,\mathrm{d}t, \tag{3.10b}$$

where $\hat{F}_3$, $\hat{F}_4$ and $\hat{F}_5$ are constants and the no-slip conditions $u_{301} = \mathrm{d}u_{301}/\mathrm{d}\eta = v_{301} = 0$ on $\eta = 0$ have been applied. Clearly, as we pass to the limit $\eta \rightarrow \infty$ and into the central region in which $\xi = \eta(2/\sigma)^{\frac{1}{2}} = O(1)$, the interaction terms I_{301} and J_{301} will play roles in determining the matching conditions between the two zones. We shall address this problem in §3.1 but first we complete the analysis required to obtain the evolution equations for the Stokes modes.

In (3.5) we presented the governing system for the leading-order Stokes mode quantities $u_{1(r)0}$ and $v_{1(r)0}$. Since we have chosen to study the problem in which the two short-wavelength modes are themselves nonlinear, we need to obtain the governing equations for the first harmonic terms $(u_{21(r)0}, v_{21(r)0})$ and the mean flow distortion terms $(u_{m(r)0}, v_{m(r)0})$ in expansions (3.3). Since the two Stokes modes have the same leading-order wavenumber m_c, it follows that the functions corresponding to the zeroth-order first harmonic terms of the second mode are the same as those for the first mode. Additionally, the same is true for the functions at leading order in the 'mixed-mode' term $E_1 E_2$ in (3.1) so that

$$u_{21(r)0}(\eta) = u_{22(r)0}(\eta) = u_{12(r)0}(\eta), \quad v_{21(r)0}(\eta) = v_{22(r)0}(\eta) = v_{12(r)0}(\eta), \quad r = 0, \pm 1, \dots. \tag{3.11}$$

Following the approach of Seminara & Hall (1977) we find that the functions $(u_{21(r)0}, v_{21(r)0})$ satisfy the systems

$$\begin{aligned}(\mathrm{L}_2 - 4\mathrm{i}r)\,\mathrm{L}_2\, u_{21(2r)0} - 2m_c^2 T_0[\mathrm{e}^{-\eta(1+\mathrm{i})} v_{21(2r-1)0} + \mathrm{e}^{-\eta(1-\mathrm{i})} v_{21(2r+1)0}] \\ = 4m_c^2 T_0 \sum_{k=-\infty}^{\infty} v_{1(2k-1)0}\, v_{1(2r-2k+1)0} + 8 \sum_{k=-\infty}^{\infty} \left(\frac{\mathrm{d}^3 u_{1(2k)0}}{\mathrm{d}\eta^3} u_{1(2r-2k)0} - \frac{\mathrm{d}u_{1(2k)0}}{\mathrm{d}\eta} \frac{\mathrm{d}^2 u_{1(2r-2k)0}}{\mathrm{d}\eta^2} \right),\end{aligned} \tag{3.12a}$$

$$\begin{aligned}(\mathrm{L}_2 - 4\mathrm{i}r + 2\mathrm{i})\, v_{21(2r-1)0} + (1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})} u_{21(2r-2)0} + (1-\mathrm{i})\,\mathrm{e}^{-\eta(1-\mathrm{i})} u_{21(2r)0} \\ = 4 \sum_{k=-\infty}^{\infty} \left(u_{1(2k)0} \frac{\mathrm{d}v_{1(2r-2k-1)0}}{\mathrm{d}\eta} - \frac{\mathrm{d}u_{1(2k)0}}{\mathrm{d}\eta} v_{1(2r-2k-1)0} \right),\end{aligned} \tag{3.12b}$$

with $$u_{21(2r)0} = \frac{\mathrm{d}u_{21(2r)0}}{\mathrm{d}\eta} = v_{21(2r-1)0} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \rightarrow \infty,$$

$$\mathrm{L}_2 \equiv \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - 4m_c^2 \quad \text{and} \quad u_{21(2s+1)0} = v_{21(2s)0} \equiv 0, \quad s = 0, \pm 1, \pm 2, \dots.$$

Further, the zeroth-order mean-flow distortion terms are given by

$$u_{m(r)0} = v_{m(2r)0} \equiv 0, \quad r = 0, \pm 1, \pm 2, \dots,$$

and

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - 4\mathrm{i}r + 2\mathrm{i} \right) v_{m(2r-1)0} = 4 \sum_{k=-\infty}^{\infty} \left(u_{1(2k)0} \frac{\mathrm{d}v_{1(2r-2k-1)0}}{\mathrm{d}\eta} + \frac{\mathrm{d}u_{1(2k)0}}{\mathrm{d}\eta} v_{1(2r-2k-1)0} \right), \tag{3.13}$$

with $v_{m(2r-1)0} = 0$ on $\eta = 0$ and as $\eta \rightarrow \infty$.

On substituting expansions (3.3) into (2.3) and equating coefficients of E_1 at descending powers of σ, it is found that terms arising from the interaction of the long-wavelength mode and the second Stokes mode appear at seventh order; i.e. in the equations determining $u_{1(r)7}$ and $v_{1(r)7}$ of (3.3b). Up to this point the successive systems of governing differential equations for $(u_{1(r)j}, v_{1(r)j};\ j = 3, 4, 5, 6)$ with appropriate boundary conditions at $\eta = 0$ and as $\eta \to \infty$ are eigenvalue problems for the constants $a_2, a_3, \ldots$ in (2.5a) and for the constants T_1 and T_2 in the expansion of the Taylor number in (2.6a). Fortunately, the values of these parameters are not explicitly required in order to compute the evolution equations and so we consider immediately the system which determines the seventh-order coefficients for the first Stokes mode in (3.3b). After simplification, the governing equations may be written as

$$u_{1(2r+1)7} = v_{1(2r)7} = 0, \quad r = 0, \pm 1, \pm 2, \ldots, \tag{3.14a}$$

$$\begin{aligned}(\mathrm{L}-4\mathrm{i}r)\,\mathrm{L}u_{1(2r)7} - \tfrac{1}{2}T_0\, m_c^2[\mathrm{e}^{-\eta(1+\mathrm{i})}v_{1(2r-1)7} + \mathrm{e}^{-\eta(1-\mathrm{i})}v_{1(2r+1)7}] \\ = (S_{2r})\frac{\mathrm{d}A}{\mathrm{d}\tau} + m_c^2(T_3 - \hat{T}_3)(P_{2r})A + (m_c^2\, T_0\, Q_{2r} + R_{2r})A(|A|^2+|B|^2) + (N_{2r})BC,\end{aligned} \tag{3.14b}$$

$$\begin{aligned}(\mathrm{L}-4\mathrm{i}r-2\mathrm{i})\,v_{1(2r+1)7} + (1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})}u_{1(2r)7} + (1-\mathrm{i})\,\mathrm{e}^{-\eta(1-\mathrm{i})}u_{1(2r+2)7} \\ = 2v_{1(2r+1)0}\frac{\mathrm{d}A}{\mathrm{d}\tau} + (Z_{2r+1})A(|A|^2+|B|^2) + (Y_{2r+1})BC,\end{aligned} \tag{3.14c}$$

with boundary conditions

$$u_{1(2r)7} = \frac{\mathrm{d}u_{1(2r)7}}{\mathrm{d}\eta} = v_{1(2r+1)7} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty, \tag{3.14d}$$

and where the coefficients $P_{2r}, Q_{2r}, R_{2r}, S_{2r}, N_{2r}, Z_{2r+1}, Y_{2r+1}$ are functions of η defined by

$$P_{2r} = \tfrac{1}{2}(v_{1(2r-1)0}\,\mathrm{e}^{-\eta(1+\mathrm{i})} + v_{1(2r+1)0}\,\mathrm{e}^{-\eta(1-\mathrm{i})}), \tag{3.15a}$$

$$Q_{2r} = 2\sum_{k=-\infty}^{\infty}(v_{m(2k+1)0}\,v_{1(2r-2k-1)0} + v_{12(2k+1)0}\,v_{1(2k-2r+1)0}), \tag{3.15b}$$

$$\begin{aligned}R_{2r} = 4\sum_{k=-\infty}^{\infty}\Bigg(&\frac{\mathrm{d}^2u_{12(2k)0}}{\mathrm{d}\eta^2}\frac{\mathrm{d}u_{1(2r-2k)0}}{\mathrm{d}\eta} + \frac{1}{2}\frac{\mathrm{d}^3u_{12(2k)0}}{\mathrm{d}\eta^3}u_{1(2r-2k)0} \\ &- u_{12(2k)0}\frac{\mathrm{d}^3u_{1(2r-2k)0}}{\mathrm{d}\eta^3} - \frac{1}{2}\frac{\mathrm{d}u_{12(2k)0}}{\mathrm{d}\eta}\frac{\mathrm{d}^2u_{1(2r-2k)0}}{\mathrm{d}\eta^2}\Bigg) \\ &- 4m_c^2\sum_{k=-\infty}^{\infty}\left(2u_{12(2k)0}\frac{\mathrm{d}u_{1(2r-2k)0}}{\mathrm{d}\eta} + \frac{3}{2}\frac{\mathrm{d}u_{12(2k)0}}{\mathrm{d}\eta}u_{1(2r-2k)0}\right),\end{aligned} \tag{3.15c}$$

$$S_{2r} = 2\left(\frac{\mathrm{d}^2u_{1(2r)0}}{\mathrm{d}\eta^2} - m_c^2\,u_{1(2r)0}\right), \tag{3.15d}$$

$$N_{2r} = \frac{m_c^3}{a_1}u_{2(2r)0}\frac{\mathrm{d}u_{300}}{\mathrm{d}\eta} + \frac{m_c}{2a_1}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(u_{2(2r)0}\frac{\mathrm{d}^2u_{300}}{\mathrm{d}\eta^2} - \frac{\mathrm{d}u_{2(2r)0}}{\mathrm{d}\eta}\frac{\mathrm{d}u_{300}}{\mathrm{d}\eta}\right), \tag{3.15e}$$

$$\begin{aligned}Z_{2r+1} = 4\sum_{k=-\infty}^{\infty}\Bigg(&u_{12(2k)0}\frac{\mathrm{d}v_{1(2r-2k+1)0}}{\mathrm{d}\eta} + \frac{\mathrm{d}v_{12(2k+1)0}}{\mathrm{d}\eta}u_{1(2r-2k)0} \\ &+ \frac{1}{2}\frac{\mathrm{d}u_{1(2k)0}}{\mathrm{d}\eta}v_{1(2r-2k+1)0} + v_{12(2k+1)0}\frac{\mathrm{d}u_{1(2r-2k)0}}{\mathrm{d}\eta}\Bigg),\end{aligned} \tag{3.15f}$$

and

$$Y_{2r+1} = -\frac{m_c}{a_1} v_{2(2r+1)0} \frac{du_{300}}{d\eta}. \tag{3.15g}$$

In (3.14) the constant $\hat{T}_3$ is that value of T_3 for which a linear vortex with z-dependence given by E_1 would be neutrally stable if the second Stokes mode and the long-wavelength mode were absent. However, the actual value of $\hat{T}_3$ need not be evaluated here.

Since the differential operator forms in the system (3.14), (3.15) are identical to those in (3.5), the above equations for $u_{1(2r)7}, v_{1(2r+1)7}$ only have a solution if a certain compatibility requirement is met, and this yields directly the desired evolution equations for the first Stokes mode. This condition is derived by considering the system adjoint to (3.5) which is given by the functions $\{F^+_{2r}(\eta), G^+_{2r+1}(\eta), r = 0, \pm 1, \ldots\}$, where

$$(\mathrm{L}+4\mathrm{i}r) LF^+_{2r} + (1-\mathrm{i})\,\mathrm{e}^{-\eta(1-\mathrm{i})} G^+_{2r+1} + (1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})} G^+_{2r-1} = 0, \tag{3.16a}$$

$$(\mathrm{L}+4\mathrm{i}r+2\mathrm{i})\, G^+_{2r+1} - \frac{m_c^2 T_0}{2}[\mathrm{e}^{-\eta(1+\mathrm{i})} F^+_{2r} + \mathrm{e}^{-\eta(1-\mathrm{i})} F^+_{2r+2}] = 0, \tag{3.16b}$$

with boundary conditions

$$\left.\begin{aligned} F^+_{2r} = \frac{dF^+_{2r}}{d\eta} = G^+_{2r+1} = 0 \quad &\text{on} \quad \eta = 0, \\ F^+_{2r}, \frac{dF^+_{2r}}{d\eta}, G^+_{2r+1}, \to 0 \quad &\text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{3.16c}$$

We multiply (3.14*b*) by the function F^+_{2r}, (3.14*c*) by G^+_{2r+1}, sum the resulting equations over all integers r and finally integrate by parts twice. On applying boundary conditions (3.14*d*) and (3.16*c*) we obtain the evolution equation

$$\frac{dA}{d\tau} = \tilde{a}_1(T_3 - \hat{T}_3)A + \tilde{a}_2 A(|A|^2 + |B|^2) + \tilde{a}_3 BC, \tag{3.17}$$

where the constants $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ are given by

$$\tilde{a}_1 = -m_c^2 \int_0^\infty \frac{\sum\limits_{r=-\infty}^{\infty} P_{2r} F^+_{2r}}{\sum\limits_{r=-\infty}^{\infty} (S_{2r} F^+_{2r} + 2v_{1(2r+1)0}\, G^+_{2r+1})} d\eta, \tag{3.18a}$$

$$\tilde{a}_2 = -\int_0^\infty \frac{\sum\limits_{r=-\infty}^{\infty} [(m_c^2 T_0 Q_{2r} + R_{2r}) F^+_{2r} + Z_{2r+1}\, G^+_{2r+1}]}{\sum\limits_{r=-\infty}^{\infty} (S_{2r} F^+_{2r} + 2v_{1(2r+1)0}\, G^+_{2r+1})} d\eta, \tag{3.18b}$$

and

$$\tilde{a}_3 = \int_0^\infty \frac{\sum\limits_{r=-\infty}^{\infty} (N_{2r} F^+_{2r} + Y_{2r+1}\, G^+_{2r+1})}{\sum\limits_{r=-\infty}^{\infty} (S_{2r} F^+_{2r} + 2v_{1(2r+1)0}\, G^+_{2r+1})} d\eta. \tag{3.18c}$$

We can repeat all the work described above to determine the amplitude equation for the second Stokes mode (with z-dependence E_2 and amplitude $B(\tau)$). This procedure is almost identical to that given previously and we find that

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = \tilde{a}_1(T_3-\hat{T}_3)B+\tilde{a}_2 B(|A|^2+|B|^2)+\tilde{a}_3 A\bar{C}, \tag{3.19}$$

where the coefficients in this equation are as given in (3.18).

To complete the triad of evolution equations we need to consider the perturbation quantities for the long-wavelength mode inside the central region. It is easily shown by examining the systems of equations for the Stokes modes in the region $\eta = O(1)$ that these short-wavelength vortices decay exponentially as $\eta \to \infty$. Consequently, in the region where $\xi \equiv \eta(2/\sigma)^{\frac{1}{2}} = O(1)$ only the long-wavelength vortex is present and we now consider this zone.

3.1. *The central region*

In this layer, where $\xi = \eta(2/\sigma)^{\frac{1}{2}}$, $0 < \xi < 1$, (2.2) implies that the basic velocity profile $\bar{u}_0$ is steady and in particular

$$\bar{u}_0 = \frac{1}{\epsilon}(1-\xi). \tag{3.20}$$

Further, the governing equations (2.3) for perturbation velocities $(\hat{u}, \hat{v}, \hat{w})$ become

$$\left(\mathrm{N}-2\frac{\partial}{\partial t}\right)\mathrm{N}\hat{u}+T\bar{u}_0\frac{\partial^2\hat{v}}{\partial z^2} = \frac{\partial^2\hat{Q}_1}{\partial z^2}-\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\frac{\partial^2\hat{Q}_2}{\partial\xi\,\partial z}, \tag{3.21a}$$

$$\left(\mathrm{N}-2\frac{\partial}{\partial t}\right)\hat{v}-2\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\hat{u}\frac{\mathrm{d}\bar{u}_0}{\mathrm{d}\xi} = \hat{Q}_3, \tag{3.21b}$$

$$\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\frac{\partial\hat{u}}{\partial\xi}+\frac{\partial\hat{w}}{\partial z} = 0, \tag{3.21c}$$

where $\mathrm{N} \equiv (2/\sigma)(\partial^2/\partial\xi^2)+(\partial^2/\partial z^2)$ and the nonlinear terms $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$ are given by

$$\hat{Q}_1 = 2\left(\hat{u}\frac{\partial\hat{u}}{\partial\xi}\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}+\hat{w}\frac{\partial\hat{u}}{\partial z}\right)-\tfrac{1}{2}T\hat{v}^2, \tag{3.21d}$$

$$\hat{Q}_2 = 2\left(\hat{u}\frac{\partial\hat{w}}{\partial\xi}\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}+\hat{w}\frac{\partial\hat{u}}{\partial z}\right), \tag{3.21e}$$

and

$$\hat{Q}_3 = 2\left(\hat{u}\frac{\partial\hat{v}}{\partial\xi}\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}+\hat{w}\frac{\partial\hat{v}}{\partial z}\right). \tag{3.21f}$$

We know that in this central zone only the long-wave vortex persists and it is essentially linear in character. Further, Hall has shown in H that the time-dependent parts of this disturbance are asymptotically smaller than the steady parts in the present case of $\sigma \gg 1$ and, as already mentioned, the unsteady components do not play any role in the determination of the evolution equations. Consequently, it is sufficient to concentrate on the steady part of the perturbation and using (3.1), (3.3), (3.8) and (3.9) the *time-independent* disturbance velocities in this central region develop according to the form

$$u_{\text{steady}} = \sigma^{-\frac{3}{2}}(\hat{u}_3(\sigma,\xi), \sigma^{-\frac{1}{4}}\hat{v}_3(\sigma,\xi), \hat{w}_3(\sigma,\xi))\, C(\tau)\, E_3 + \text{c.c.}, \tag{3.22a}$$

and we write
$$(\hat{u}_3, \hat{v}_3, \hat{w}_3) = \sum_{j=0}^{2} \sigma^{-\frac{1}{4}j}(\hat{u}_{30j}(\xi), \hat{v}_{30j}(\xi), \hat{w}_{30j}(\xi)) + o(\sigma^{-1}). \tag{3.22b}$$

Substituting (3.22) into the governing equations (3.21) and comparing leading-order terms yields the eigenproblem

$$\mathrm{N}_1^2\, \hat{u}_{300} - T_0\, a_1^2\, \epsilon_1 (1-\xi)\, \hat{v}_{300} = 0, \tag{3.23a}$$

$$\mathrm{N}_1\, \hat{v}_{300} + \surd 2\epsilon_1\, \hat{u}_{300} = 0, \tag{3.23b}$$

where $\mathrm{N}_1 \equiv (\mathrm{d}^2/\mathrm{d}\xi^2) - 2a_1^2$. Matching with the inner-layer solutions (3.8) as $\xi \to 0$ gives the boundary conditions

$$\hat{u}_{300} = \frac{\mathrm{d}\hat{u}_{300}}{\mathrm{d}\xi} = \hat{v}_{300} = 0 \quad \text{at} \quad \xi = 0. \tag{3.23c}$$

To obtain boundary conditions as $\xi \to 1$ (i.e. as the outer cylinder is approached, see figure 1), it was shown in H that, strictly, we need to consider details of the 'Stokes-like' layer adjacent to the outer cylinder. However, again fortunately, it can be demonstrated by routine, although fairly lengthy analysis as in H, that to the orders of working we shall be dealing with here, the appropriate boundary conditions at $\xi = 1$ are given by

$$\hat{u}_{30j} = \frac{\mathrm{d}\hat{u}_{30j}}{\mathrm{d}\xi} = \hat{v}_{30j} = 0, \quad j = 0, 1, 2. \tag{3.24}$$

We recall that at the outset T_0 and ϵ_1 were specifically chosen such that the 'effective' Taylor number $T_0\epsilon_1^2$ for the system (3.23), (3.24) took its critical value. This ensures that this system forms an eigenproblem for the wavenumber a_1 and the corresponding disturbance eigenfunctions $\hat{u}_{300}$ and $\hat{v}_{300}$. From H we have $a_1 = 2.2107$.

The eigensystem encountered at the next order in σ serves to determine the coefficient ϵ_2 in the expansion of the steady part of the angular velocity of the inner cylinder as defined by (2.6*b*). At this stage no effects of the interaction process appear, the calculation is routine and yields $\epsilon_2 = 0$. Conversely, we find that the unknowns $\hat{u}_{302}, \hat{v}_{302}$ satisfy

$$\mathrm{N}_1^2\, \hat{u}_{302} - T_0\, \epsilon_1\, a_1^2 (1-\xi)\, \hat{v}_{302} = \left(\frac{1}{C}\frac{\mathrm{d}C}{\mathrm{d}\tau}\right) \mathrm{N}_1\, \hat{u}_{300} + a_1^2(1-\xi)\,(T_0\,\epsilon_3 + T_1\,\epsilon_1)\,\hat{v}_{300}, \tag{3.25a}$$

$$\mathrm{N}_1\, \hat{v}_{302} + \surd 2\epsilon_1\, \hat{u}_{302} = \left(\frac{1}{C}\frac{\mathrm{d}C}{\mathrm{d}\tau}\right)\hat{v}_{300} - \surd 2\epsilon_3\, \hat{u}_{300}, \tag{3.25b}$$

and that the interaction between the two Stokes modes enters the relevant boundary conditions. At $\xi = 1$ we have that (3.24) applies, but at $\xi = 0$ we obtain from (3.3) and (3.10) that

$$\hat{u}_{302} = 0, \quad \frac{\mathrm{d}\hat{u}_{302}}{\mathrm{d}\xi} = \frac{1}{2}\int_0^\infty t^2 I_{301}(t)\,\mathrm{d}t \quad \text{and} \quad \hat{v}_{302} = \int_0^\infty t J_{301}(t)\,\mathrm{d}t. \tag{3.26}$$

As was the case for the Stokes mode problem, we derive the evolution equation for this long-wavelength mode by considering the problem adjoint to (3.23). This adjoint system is the pair $(f_0^+(\xi), g_0^+(\xi))$ where

$$\mathrm{N}_1^2 f_0^+ + \sqrt{2}\epsilon_1 g_0^+ = 0, \quad \mathrm{N}_1 g_0^+ - T_0 a_1^2 \epsilon_1 (1-\xi) f_0^+ = 0, \tag{3.27}$$

with $f_0^+ = (\mathrm{d}f_0^+/\mathrm{d}\xi) = g_0^+ = 0$ at $\xi = 0, 1$. In the usual way, on multiplying (3.25*a*) by f_0^+, (3.25*b*) by g_0^+, adding the results and integrating by parts we obtain the amplitude equation

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = c_1(\epsilon_3 - \hat{\epsilon}_3)\, C + c_2 A\bar{B}, \tag{3.28a}$$

where
$$c_1 = -\left(\frac{\displaystyle\int_0^1 (T_0 a_1^2 (1-\xi)\, \hat{v}_{300} f_0^+ - \sqrt{2}\hat{u}_{300}\, g_0^+)\, \mathrm{d}\xi}{\displaystyle\int_0^1 ((N_1\, \hat{u}_{300}) f_0^+ + \hat{v}_{300}\, g_0^+)\, \mathrm{d}\xi} \right), \tag{3.28b}$$

$$c_2 = \left(\frac{\displaystyle\left(\int_0^\infty t J_{301}^\dagger(t)\, \mathrm{d}t\right) \frac{\mathrm{d}g_0^+}{\mathrm{d}\xi}\bigg|_{\xi=0} + \left(\frac{1}{2}\int_0^\infty t^2 I_{301}^\dagger(t)\, \mathrm{d}t\right) \frac{\mathrm{d}^2 f_0^+}{\mathrm{d}\xi^2}\bigg|_{\xi=0}}{\displaystyle\int_0^1 ((N_1\, \hat{u}_{300}) f_0^+ + \hat{v}_{300}\, g_0^+)\, \mathrm{d}\xi} \right), \tag{3.28c}$$

where $\hat{\epsilon}_3$ is the value of ϵ_3 at which the long-wavelength mode is neutrally stable according to linear theory. As was the case for $\hat{T}_3$, the value of this constant need not be found explicitly. Unlike the amplitude equations (3.17) and (3.19) this third equation (3.28*a*) does not contain a cubic term. Such a term, if it were to appear, would necessarily arise from self-interactions involving the fundamental component of the long-wavelength vortex. However, we know from the nonlinear analysis of H, that such a mechanism cannot occur until the amplitude of the long-wave vortex within the central region becomes $O(\sigma^{-1})$. So, according to (3.22*a*), the long-wave disturbance is not subject to this type of nonlinearity and hence we have the absence of the cubic terms in (3.28*a*).

In summary, we now have the three coupled amplitude equations for the interacting modes, given by (3.17), (3.19) and (3.28*a*). To enable analysis of this triad the numerical values of the coefficients contained in (3.18) and (3.28*b*, *c*) are required. The determination of these constants is the subject of §4.

4. Numerical work

In order to calculate the numerical coefficients in the coupled amplitude equations it was first necessary to perform some preliminary analysis of the eigensystem (3.5). This system was first approximated by a finite number of differential equations obtained by letting $r = 0, \pm 1, \pm 2, \ldots \pm N_*$ in (3.5) for some chosen N_* and by setting $u_{1(2r)0} = v_{1(2r+1)0} = 0$ for $|r| > N_*$. The asymptotic boundary conditions as $\eta \to \infty$ were replaced by suitable conditions at some selected large value of η, say η_∞. As discussed in detail in SH, for large η a satisfactory approximation for the system (3.5) may be obtained by neglecting the centrifugal terms. This approximate system was solved using the condition of exponential decay as $\eta \to \infty$ and hence used to generate suitable asymptotic conditions for the full system at $\eta = \eta_\infty$. These conditions were used to initiate an explicit fourth-order Runge–Kutta scheme to determine the eigensolutions of (3.5) by marching with a preselected step length h from $\eta = \eta_\infty$ to $\eta = 0$. The Runge–Kutta method was combined with an iterative technique to ensure that for each wavenumber m, T_0 was determined such that the necessary boundary conditions at $\eta = 0$ were satisfied. The results obtained using this procedure were checked to ensure independence from the choices of η_∞, the integration step length h and the point at which system (3.5) was truncated, i.e. N_*. The values of the coefficients in the amplitude equations (3.28) depend on the precise normalization chosen for the solutions of (3.5) and we made the choice such that the maximum value of $u_{1(0)0}$ for $0 \leqslant \eta < \infty$ is unity. We found, in agreement with the results of SH, that the critical Taylor number was $T_0 = T_{sc} = 232.522$ and that this occurs at corresponding disturbance wavenumber $m = m_c = 0.85852$. Figures of the whole neutral stability curve in Taylor number/wavenumber space and of the eigensolutions of (3.5) in the critical case are to be found in Seminara (1976).

The same procedure was employed to calculate the eigensolutions $F^+_{2r}(\eta)$ and $G^+_{2r+1}(\eta)$ of the adjoint system (3.16c). Using these adjoint eigenfunctions together with the solutions of the fundamental system (3.5), the expressions $P_{2r}(\eta)$ and $S_{2r}(\eta)$ (defined in (3.15)) were determined, which, in turn, enabled the integrand of (3.18a) to be calculated. The coefficient $\tilde{a}_1$ was then computed by evaluating the integral (3.18a) by the trapezium rule, and the resulting value was checked for independence from the number of functions used to approximate the infinite sums present in the integrand.

To establish the integrand of (3.18b) it would be necessary to first solve the system (3.12) for the first harmonic terms and (3.13) for the mean distortion functions. Then $Q_{2r}(\eta)$, $R_{2r}(\eta)$ and $Z_{2r+1}(\eta)$ would be evaluated using definitions (3.15) and the integrand calculated. Upon numerical integration, the interaction coefficient $\tilde{a}_2$ would be obtained. Instead of carrying out this process ourselves we quote the result given by Duck (1979) who evaluated (3.18b) in the course of his investigation of nonlinear Stokes modes close to wavy cylinder walls. We note that the definitions (3.18a, b) are at variance with the equivalent ones given in Seminara & Hall (1977) owing to an error in that paper and also that the values of the coefficients $\tilde{a}_1$ and $\tilde{a}_2$ calculated by Duck differ from those quoted by Seminara & Hall. Our computations for $\tilde{a}_1$ revealed agreement with the work of Duck (1979). We additionally remark that the main conclusion of Seminara & Hall (1977) was that finite-amplitude, stable vortices could persist in the inner Stokes layer according to weakly nonlinear theory. This conclusion is only dependent upon the signs of the coefficients rather than their precise values, and even though the numerical values quoted by Seminara & Hall are incorrect, the conclusion concerning the existence of finite-amplitude vortices is unchanged.

To determine the remaining coefficient in (3.17) and (3.19) we need to calculate $\hat{u}_{300}(\xi)$ and $\hat{v}_{300}(\xi)$ which define the long-wavelength mode structure in the central region. These terms satisfy the system (3.23) and, to fix a normalization, we chose $\hat{u}_{300}, \hat{v}_{300}$ such that in addition to (3.23a–c) we had $\hat{u}''_{300}(0) = 1$. When matching between the inner layer and the central zone we find that the coefficients $\hat{F}_1$ and $\hat{F}_2$ in (3.8b) are given by $\hat{F}_1 = \frac{1}{2}$ and $\hat{F}_2 = \hat{v}'_{300}(0)$. These values are used to evaluate $N_{2r}(\eta)$ and $Y_{2r+1}(\eta)$ in (3.15) and hence the coefficient $\tilde{a}_3$. The eigenvalues of (3.23), together with the corresponding eigenfunctions, were found by Runge–Kutta and iterative techniques. As expected from H, we found that for the critical case $\epsilon_1 = 4.5407$ and $a_1 = 2.2107$ and further details of the method and sketches of $\hat{u}_{300}$ and $\hat{v}_{300}$ may be found in Hall (1973). Finally, given the functions $\hat{u}_{300}, \hat{v}_{300}$, the coefficient $\tilde{a}_3$, defined by (3.18c), was determined.

In order to calculate the coefficients c_1 and c_2 of (3.28a), the adjoint functions $(f_0^+(\xi), g_0^+(\xi))$ satisfying (3.27) were computed using a technique identical to that employed for finding $\hat{u}_{300}$ and $\hat{v}_{300}$. Then the value of c_1 followed naturally. Additionally, for c_2, the terms $I_{301}^\dagger(t)$ and $J_{301}^\dagger(t)$ had to be derived. This necessitated the solution of the infinite systems (3.6) and (3.7) using methods very similar to those described for obtaining the eigensolutions of (3.5). $I_{301}^\dagger(t)$ and $J_{301}^\dagger(t)$ followed from definitions (3.9) and the integrals in (3.28a) were again computed using the trapezium rule.

All the numerical calculations were checked for consistency and our results for the coefficients in the coupled amplitude equations (3.17), (3.19) and (3.28c) are given below. We found that

$$\left.\begin{aligned} \tilde{a}_1 = 0.00233, \quad \tilde{a}_2 = -0.0378, \quad \tilde{a}_3 = -0.135, \\ c_1 = 5.765, \quad c_2 = -0.3966\times 10^5. \end{aligned}\right\} \tag{4.1}$$

5. Results and discussion

The analysis of the previous sections has led us to conclude that the amplitudes of the two Stokes modes (A, B) and that of the long-wavelength mode (C) evolve according to the coupled triple of equations

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \tilde{a}_1(T_3-\hat{T}_3)A + \tilde{a}_2(|A|^2+|B|^2)A + \tilde{a}_3 BC, \tag{5.1a}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = \tilde{a}_1(T_3-\hat{T}_3)B + \tilde{a}_2(|A|^2+|B|^2)B + \tilde{a}_3 A\bar{C}, \tag{5.1b}$$

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = c_1(\epsilon_3-\hat{\epsilon}_3)C + c_2 A\bar{B}, \tag{5.1c}$$

where the constant (real) coefficients are given by (4.1).

A natural place to commence our analysis of (5.1) is to seek equilibrium solutions. It can be shown that there are four non-trivial steady solutions of (5.1) and if we write $A = \alpha\,\mathrm{e}^{\mathrm{i}\phi}$, $B = \beta\,\mathrm{e}^{\mathrm{i}\theta}$, $C = \gamma\,\mathrm{e}^{\mathrm{i}(\phi-\theta)}$, where α, β and γ are real, then the equilibrium solutions are given by

(i) $$\alpha = 0, \quad \gamma = 0, \quad \beta = \pm f_1(T_3-\hat{T}_3), \tag{5.2a}$$

(ii) $$\beta = 0, \quad \gamma = 0, \quad \alpha = \pm f_1(T_3-\hat{T}_3), \tag{5.2b}$$

(iii) $$\alpha = \beta = \pm f_2(T_3-\hat{T}_3, \epsilon_3-\hat{\epsilon}_3), \quad \gamma = \pm f_3(T_3-\hat{T}_3, \epsilon_3-\hat{\epsilon}_3), \tag{5.2c}$$

(iv) $$\alpha = -\beta = \pm f_2(T_3-\hat{T}_3, \epsilon_3-\hat{\epsilon}_3), \quad \gamma = \mp f_3(T_3-\hat{T}_3, \epsilon_3-\hat{\epsilon}_3), \tag{5.2d}$$

where f_1, f_2 and f_3 are defined by

$$f_1(x) = \left(\frac{-\tilde{a}_1 x}{\tilde{a}_2}\right)^{\frac{1}{2}}, \tag{5.2e}$$

$$f_2(x,y) = \left(\frac{\tilde{a}_1 c_1 xy}{\tilde{a}_3 c_2 - 2\tilde{a}_2 c_1 y}\right)^{\frac{1}{2}}, \tag{5.2f}$$

and $$f_3(x,y) = \frac{-c_2\tilde{a}_1 x}{\tilde{a}_3 c_2 - 2\tilde{a}_2 c_1 y}. \tag{5.2g}$$

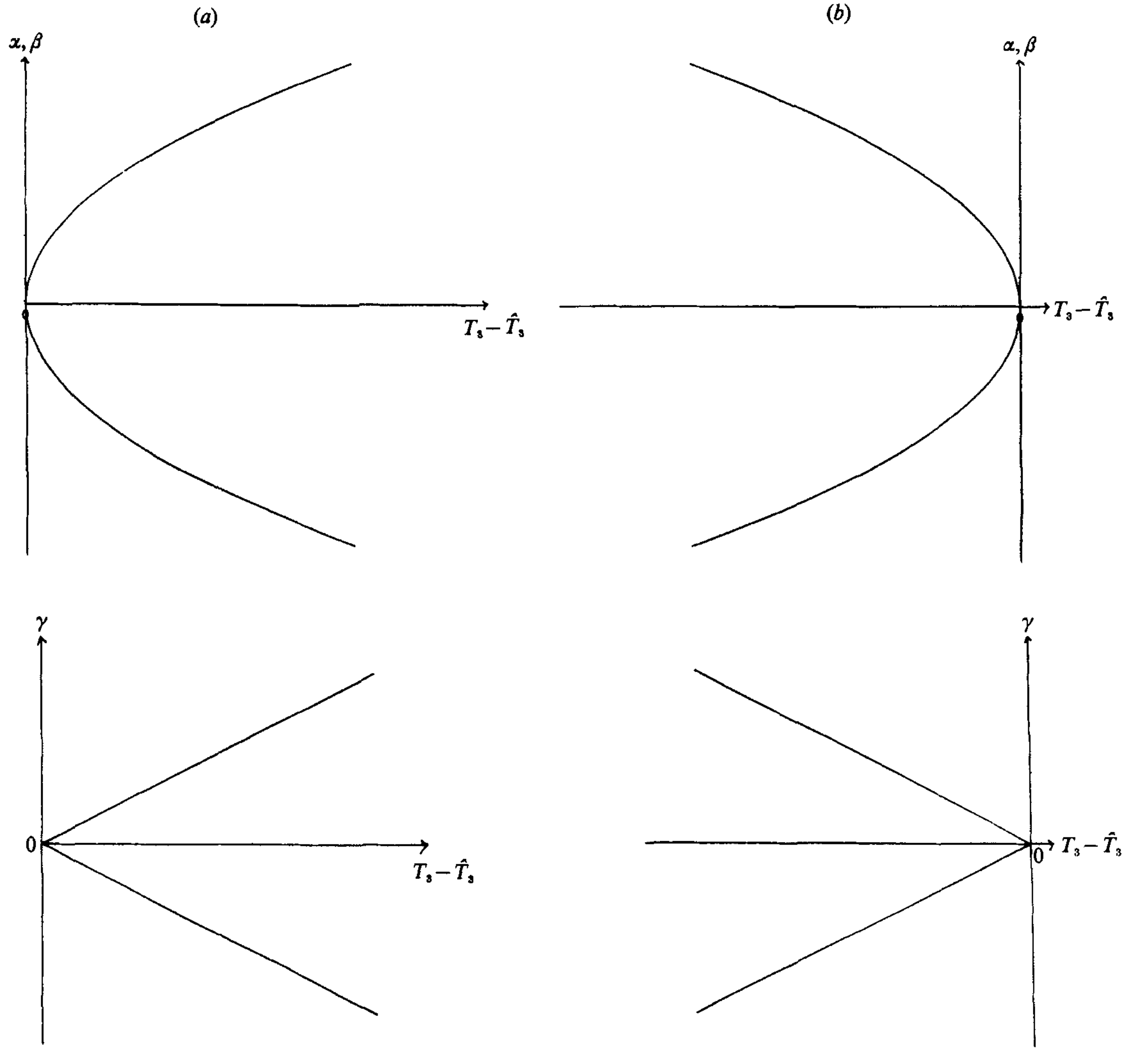

FIGURE 2. Schematic dependence of the vortex amplitudes, α, β and γ upon the parameter $(T_3 - \hat{T}_3)$ in the cases of the equilibrium solutions (5.2*c*, *d*). (*a*) $0 > \Phi > \Phi_c$, (*b*) $\Phi > 0$, $\Phi < \Phi_c$.

The solutions (i) and (ii) correspond to having only a single Stokes mode present in the flow. Thus (5.2*a*, *b*) are just the finite-amplitude equilibrium vortices predicted by the weakly nonlinear theory of Seminara & Hall (1977), who reported that this theory is consistent with observations made in their experiments. The third and fourth solutions occur when all three modes of instability are present in the fluid. These finite-amplitude steady vortices can only exist in certain regions of (ϵ_3, T_3) parameter space. In particular, when

$$\Phi \equiv (\epsilon_3 - \hat{\epsilon}_3) > \Phi_c = \frac{\tilde{a}_3 c_2}{2\tilde{a}_2 c_1} \approx -1.23 \times 10^4,$$

then (5.2*f*) implies that we must have $(T_3 - \hat{T}_3) < 0$ if $0 > \Phi > \Phi_c$ and $(T_3 - \hat{T}_3) > 0$ otherwise. In all cases, for a fixed Φ, we find that α and β are proportional to the square root of $|T_3-\hat{T}_3|$ whereas γ is a linear multiple of $|T_3-\hat{T}_3|$. Sketches of the possible equilibrium amplitude configurations are given in figure 2. A simple linear stability analysis of these equilibrium solutions reveals that when $(\epsilon_3-\hat{\epsilon}_3)>0$ then the vortices are unstable whereas for $(\epsilon_3-\hat{\epsilon}_3)<0$ they are stable. This was also confirmed using a numerical integration of the triad of equations (5.1). Hence, we would expect that, experimentally, for apparatus tuned such that the long-wavelength vortex is linearly stable when isolated from the other modes, we may hope to be able to observe a stable configuration in which all three vortex modes are present. Whether this effect could possibly be observed in practice remains a matter of speculation for the present although to the best of the authors' knowledge there is no direct experimental evidence with which to compare our results. Also in an experimental setting, the role that end effects might play is significant. In our theoretical analysis no account has been taken of the obvious physical requirement that in practice the cylinders would be of finite length, and the incorporation of this restriction provides scope for an extension of our work.

In general the solution of (5.1) is a non-trivial matter but some specific cases may give insight into behaviour of the solutions.

Let us consider the situation where small-amplitude disturbances develop over a slow timescale. We assume that the perturbation of the Taylor number and ϵ are also small so that all these quantities are scaled on the same parameter $\delta, \delta \ll 1$. Hence,

$$(A,B,C)=\delta(a,b,c), \quad \hat{\tau}=\delta\tau,$$

$$(T_3-\hat{T}_3, \epsilon_3-\hat{\epsilon}_3)=\delta(t_3-\hat{t}_3, e_3-\hat{e}_3).$$

Substituting these into the system of equations (5.1) and then neglecting the cubic terms that are $O(\delta)$ smaller than the others, we have

$$\frac{\mathrm{d}a}{\mathrm{d}\hat{\tau}}=\tilde{a}_1(t_3-\hat{t}_3)\,a+\tilde{a}_3\,bc, \tag{5.3a}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\hat{\tau}}=\tilde{a}_1(t_3-\hat{t}_3)\,b+\tilde{a}_3\,a\bar{c}, \tag{5.3b}$$

$$\frac{\mathrm{d}c}{\mathrm{d}\hat{\tau}}=c_1(e_3-\hat{e}_3)\,c+c_2\,a\bar{b}. \tag{5.3c}$$

Again, we look for equilibrium solutions of these reduced equations by writing $a=\alpha\,\mathrm{e}^{\mathrm{i}\phi}$, $b=\beta\,\mathrm{e}^{\mathrm{i}\theta}$, $c=\gamma\,\mathrm{e}^{\mathrm{i}(\phi-\theta)}$, where α, β and γ are real. We find that the solutions are given by (5.2c) and (5.2d) with $\tilde{a}_2=0$ as would be expected. However, these solutions exist only in the regions of the parameter space where $(t_3-\hat{t}_3)(e_3-\hat{e}_3)>0$. Analysis of the stability of these equilibrium solutions shows that both (i) and (ii) are unstable for all values of the parameters $(t_3-\hat{t}_3)$ and $(e_3-\hat{e}_3)$, within the regime of existence, in contrast to the full equations (5.1). Hence, for this small-amplitude disturbance condition there is no stable configuration.

If we make further transformations

$$b_1=(c_2\,\tilde{a}_3)^{\frac{1}{2}}\,a, \quad b_2=(c_2\,\tilde{a}_3)^{\frac{1}{2}}\,\bar{b}, \quad b_3=\tilde{a}_3\,c,$$

we obtain

$$\frac{\mathrm{d}b_1}{\mathrm{d}\hat{\tau}} = \tilde{a}_1(t_3-\hat{t}_3)\,b_1+\bar{b}_2\,b_3, \tag{5.4a}$$

$$\frac{\mathrm{d}b_2}{\mathrm{d}\hat{\tau}} = \tilde{a}_1(t_3-\hat{t}_3)\,b_2+\bar{b}_1\,b_3, \tag{5.4b}$$

$$\frac{\mathrm{d}b_3}{\mathrm{d}\hat{\tau}} = c_1(e_3-\hat{e}_3)\,b_3+b_1\,b_2. \tag{5.4c}$$

Many forms of three-wave resonance are studied in detail in Craik (1985) and a particular solution is given for this system (5.4) when we have the additional constraint

$$t_3-\hat{t}_3 = -\frac{c_1}{2\tilde{a}_1}(e_3-\hat{e}_3). \tag{5.5}$$

The solution is given by

$$b_{1,2} = \mathrm{i}\sqrt{2}\,\frac{(t_3-\hat{t}_3)}{\cos\Theta}\exp\left(\mathrm{i}(\Theta_{1,2}-\tilde{a}_1(t_3-\hat{t}_3)\,\hat{\tau}\tan\Theta)\right), \tag{5.6a}$$

$$b_3 = -\tilde{a}_1\frac{(t_3-\hat{t}_3)}{\cos\Theta}\exp\left(\mathrm{i}(\Theta_3-2\tilde{a}_1(t_3-\hat{t}_3)\,\hat{\tau}\tan\Theta)\right), \tag{5.6b}$$

where the real phases Θ_j $(j = 1,2,3)$ satisfy

$$\Theta_3-\Theta_2-\Theta_1 = \Theta.$$

From (5.5) we see that the permitted values of the parameters $(t_3-\hat{t}_3)$ and $(e_3-\hat{e}_3)$ are not within the range for which the equilibrium solutions exist. Further, we find that the solutions (5.6*a*), (5.6*b*) exhibit an oscillatory behaviour as depicted in figure 3.

As has been kindly pointed out to us by one of the referees, there exist another exact solution of the system (5.4), as discussed in Craik (1985). This occurs when all three waves have the same linear growth or decay rates, i.e. when

$$\tilde{a}_1(t_3-\hat{t}_3) = c_1(e_3-\hat{e}_3) = \chi,$$

where χ is some real constant. Then if we make the transformations

$$B_j = b_j\exp(-\chi\hat{\tau}), \quad (j = 1,2,3), \qquad \hat{t} = \chi^{-1}(\exp(\chi\hat{\tau})-1),$$

we obtain the reduced system of equations

$$\frac{\mathrm{d}B_1}{\mathrm{d}\hat{t}} = B_3\bar{B}_2, \quad \frac{\mathrm{d}B_2}{\mathrm{d}\hat{t}} = B_3\bar{B}_1, \quad \frac{\mathrm{d}B_3}{\mathrm{d}\hat{t}} = B_1B_2. \tag{5.7}$$

This system gives rise to a class of solutions of the form

$$B_1 = \hat{B}\exp(\mathrm{i}\theta_1), \quad B_2 = \hat{B}\exp(\mathrm{i}\theta_2), \quad B_3 = \hat{B}\exp(\mathrm{i}\theta_3), \tag{5.8a}$$

where

$$\hat{B}(\hat{t}) = \frac{\kappa}{1-\hat{t}\kappa}, \tag{5.8b}$$

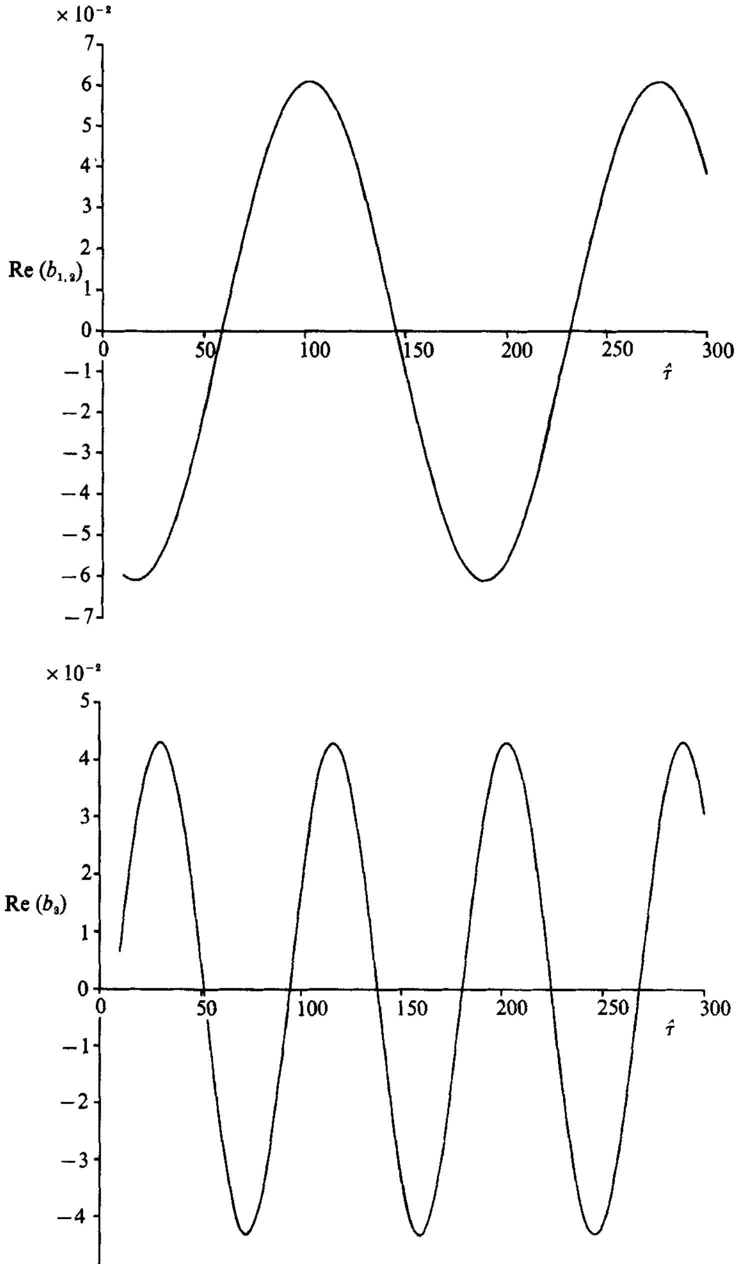

FIGURE 3. Showing the oscillatory time dependence of the solutions (5.6a, b) of the small-amplitude case for $t_3 - \hat{t}_3 = 10$, and $\Theta_1 = \Theta_2 = \Theta_3 = 1$.

the constant phases θ_j are related by

$$\theta_3 = \theta_1 + \theta_2,$$

and where κ is an arbitrary positive constant. Thus, these solutions are subject to a finite-time breakdown when $\hat{t} = \kappa^{-1}$, corresponding to singularities in b_j, $(j = 1, 2, 3)$, at $\hat{\tau} = \chi^{-1} \ln (1 + \chi/\kappa)$, provided that $\chi > -\kappa$. However, if $\chi < -\kappa$, then the linear damping is sufficiently strong to inhibit the breakdown mechanism and the waves merely decay as $\hat{\tau} \to \infty$.

Another particular case for which an analytical solution can be found is that in which the long-wave mode has an amplitude much larger than those of the short-wave vortices. Also, for simplicity, we consider the two short-wave amplitudes to be identical, which forces the amplitude of the long-wave mode to be purely real.

Suppose that the short-wave amplitudes A and B are scaled on a small parameter $\delta, \delta \ll 1$ then $(A, B) = \delta(\hat{A}, \hat{B})$ whilst C remains $O(1)$, and write $\hat{A} = \hat{B} = a\,\mathrm{e}^{\mathrm{i}\phi}$ where a and ϕ are real. Substituting into (5.1) produces

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = \tilde{a}_1(T_3 - \hat{T}_3)\,a + \tilde{a}_3\,ac,$$

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = c_1(\epsilon_3 - \hat{\epsilon}_3)\,C,$$

having neglected terms $O(\delta)$.

Solving this gives

$$a = K_2 \exp{(\mathrm{i}\phi)} \exp\left(\tilde{a}_1(T_3 - \hat{T}_3)\,\tau + \frac{\tilde{a}_3 K_1}{c_1(\epsilon_3 - \hat{\epsilon}_3)} \exp{(c_1(\epsilon_3 - \hat{\epsilon}_3)\,\tau)}\right), \tag{5.9a}$$

$$C = K_1 \exp{((\epsilon_3 - \hat{\epsilon}_3)\,\tau)}, \tag{5.9b}$$

where K_1 and K_2 are constants.

Clearly, if $T_3 - \hat{T}_3, \epsilon_3 - \hat{\epsilon}_3 < 0$ the amplitudes decay to the stable equilibrium point $a = C = 0$. However, if $T_3 - \hat{T}_3 > 0, \epsilon_3 - \hat{\epsilon}_3 < 0$ then we have the case where the long-wave mode decays exponentially but the short-wave modes grow likewise. Therefore at some point in time the initial assumption regarding the relative amplitude sizes becomes invalid. In particular if $\Phi < \Phi_c$ then the amplitudes adjust until we reach a stable equilibrium solution of the form (5.2*c*) or (5.2*d*).

Another avenue for analysis of the system (5.1) is to investigate the possibilities of either finite-time or infinite-time singularities in solutions. Such breakdowns have been found in a variety of interaction phenomena, and Hall & Smith (1988) have discovered that both types of singularity are possible in the context of interaction between vortices and Tollmien–Schlichting waves. However, apart from the specialized case in which the three modes all have the same linear growth or decay rates and for which the solution is given by (5.8), we found the singularities in the solutions of (5.1) tend to be of a logarithmic nature, do not occur as time advances and so are of no practical significance.

To conclude, we remark that we have formulated the problem in which three vortices can coexist in concentric cylinder flows for a suitable choice of parameters and interact to form a resonant triad in which two of the modes are nonlinear. Coupled evolution equations for the modes have been derived and some simple analysis of equilibrium solutions has been conducted. Some possible future lines of investigation have been suggested, although it is likely that a full description of the variety of behaviours which may be exhibited by the solutions of the triad equations would be most easily gleaned by conducting a full numerical simulation of the problem. The system (5.1) could be approximated using an appropriate finite-difference estimate for the time derivative. The system would then be marched forward in time from a selection of initial conditions and the evolution of the amplitudes monitored. However, the system (5.1) is of a sufficiently complicated structure that complete description of its properties remains a huge unresolved topic with much work to be done before overall satisfactory conclusions become feasible.

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